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An unloading wave in a medium with variable strain-wave velocities☆

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Abstract

A method is proposed for obtaining the equations of an unloading wave in the form of an algebraic polynomial in problems of the dynamics of an elastoplastic rod, in which strain waves propagate with variable velocities both when the stress increases and when it decreases.

The phenomenon of the interaction of strain waves, produced by the unloading of a rod, with strain waves which arise when it is loaded, discovered by Rakhmatulin and called by him an unloading wave,[1] does not belong to any of the classical boundary-value problems of the equations of mathematical physics. Below we propose an algorithm for solving the problem of an unloading wave for materials with non-linear physical relations, as a development of previously elaborated methods.[2–5]

1. Formulation of the problem

A semi-infinite rod of constant cross section is subjected at the end to the brief action of a dynamic load in the form of a normal stress, which gives rise to a transient process of the propagation of elastoplastic strain waves in the rod. The stress state of the rod is assumed to be one-dimensional. The transverse strain energy is ignored. The strains in the rod are assumed to be fairly small, so that we can use the Cauchy relation as a measure of the strain.

1.1. Physical relations

The Prandtl diagram defines the relation between the stress σ and the strain ε using three constants: the normal elasticity modulus E, the strengthening factor α $(0 < \alpha < 1)$ and the elastic limit ε_0 and is a graph of a piecewise-linear function:

$\sigma = \varepsilon E$ when $\varepsilon \in [0, \varepsilon_0]$ increases and decreases;
$\sigma = [\varepsilon_0 + (\varepsilon - \varepsilon_0)\alpha]E$ when the values of $\varepsilon > \varepsilon_0$ increases;
$\sigma = [\varepsilon - (e - \varepsilon_0)(1 - \alpha)]E$ when the value of ε decreases from the strain $e > \varepsilon_0$ reached during loading.

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In the problems solved below, while continuing to observe Hooke's law $\sigma=\varepsilon E$ in the strain section $[0, \varepsilon_0]$, when $\varepsilon>\varepsilon_0$ we will take $\alpha a(\varepsilon)$ and $Eb(\varepsilon)$ instead of α and E:

when $\varepsilon>\varepsilon_0$ increases

$$\sigma = [\varepsilon_0+(\varepsilon-\varepsilon_0)\alpha a(\varepsilon)]Eb(\varepsilon) \tag{1.1}$$

when the values of ε decrease from the strain $e>\varepsilon_0$ reached during loading

$$\sigma = \{\varepsilon-(e-\varepsilon_0)[1-\alpha a(e)]\}Eb(e) \tag{1.2}$$

It was shown in Ref. 6 that the functions $a(\varepsilon)$ and $b(\varepsilon)$ can be interpreted as quantities which represent the increase in the degree of disintegration of the material. To fix our ideas, in order to be able to solve the direct problem of an unloading wave, we will assume

$$a(\varepsilon) = \frac{\beta}{\beta-1+\varepsilon/\varepsilon_0}, \quad b(\varepsilon) = 1-\left(\frac{\varepsilon/\varepsilon_0-1}{\beta}\right)^2 \tag{1.3}$$

where $\beta=(e_*-\varepsilon_0)/\varepsilon_0$ and e_* is the limit of deformability of the material. The diagrams of $\sigma(\varepsilon)$ then have the form characteristic for certain materials.

What is new in the assumed model of the medium is the fact that the slope of the unloading line of the $\sigma(\varepsilon)$ diagram outside the elastic limit ε_0 decreases as the maximum strain reached during loading increases. This effect, observed experimentally quite long ago (see, for example, Ref. 7) has not previously been taken into account in the problem of an unloading wave.

1.2. *The resolving system of equations*

We will direct the axis of the rod along the positive semiaxis of the spatial variable x. The resolving system of equations of the problem, if we take the normal stress $\sigma=\sigma(x,t)$ in the cross section of the rod and the velocity of displacement $v=v(x,t)$ of the cross section in the direction of the x axis as the unknowns, has the form

$$\partial\sigma/\partial x-\rho\partial v/\partial t = 0, \quad \partial\sigma/\partial t-\rho z^2\partial v/\partial x = 0; \quad z = \sqrt{\sigma'(\varepsilon)/\rho} \tag{1.4}$$

The quantity z has the meaning of the propagation velocity of strain waves. If we put

$$g(\varepsilon) = \beta^{-1}\sqrt{\alpha\beta(\beta+2)+2-2(\alpha\beta+1)\varepsilon/\varepsilon_0}, \quad s(e) = \sqrt{b(e)} \tag{1.5}$$

then $z=\sqrt{E/\rho}=c$ is the velocity of sound in the rod material when $\varepsilon\in[0, \varepsilon_0]$, $z=cg(\varepsilon)$ during loading with $\varepsilon\in(\varepsilon_0, e_*)$, $z=cs(e)$ when ε is decreases from the strain $e\in(\varepsilon_0, e^*)$ reached during loading. Hence it follows that when ε increases in the range of plastic strains the velocity of the waves $z=cg(\varepsilon)$ is a function which decreases from $z=kc$ (here $k=\sqrt{\alpha}$) when $\varepsilon=\varepsilon_0$ to $z=cg(\varepsilon_m)$ (ε_m is the maximum strain reached during loading); when ε decreases from the strain e reached during loading, the velocity of the waves $z=cs(e)$ is independent of the current value of ε and is a function which decreases from $z=c$ when $e=\varepsilon_0$ to $z=cs(\varepsilon_m)$ when $e=\varepsilon_m$.

The equation $g(\varepsilon_*=0$ defines the critical strain ε_*. When $\varepsilon=\varepsilon_*$ the function $\sigma(\varepsilon)$ reaches a local maximum, while the velocity of the strain waves $z=cg(\varepsilon_*)$ is equal to zero. Wave-process problems in the rod considered can be solved in the range of strains that are less than the critical value ε_*. In the case of functions $a(\varepsilon)$ and $b(\varepsilon)$ of the form (1.3) the critical strain is given by the formula

$$\varepsilon_* = (\varepsilon_0/2)[\alpha\beta(\beta+2)+2]/(\alpha\beta+1)$$

The differential equations of the characteristics of system of equations (1.4) and the differential relations on them have the form

$$dx = \pm z dt, \quad d\sigma\mp\rho z dv = 0 \tag{1.6}$$

The load on the ends of the rod, which are equal to zero at the initial instant of time $t=0$, increases monotonically as $\sigma(0,t)=q(t)$ and reaches maximum value when $t=T$. Beginning from this instant of time, it decreases monotonically as $\sigma(0,t)=p(t)$, where $p(T)=q(T)$, and becomes equal to zero when $t=T_0$. Subsequently there is no load on the rod. We

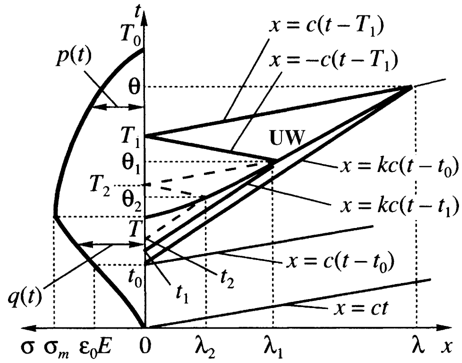

Fig. 1.

will denote by t_0 the instant of time when the stress on the end of the rod $q(t)$ as the load increases reaches the elastic limit $\sigma_0 = \varepsilon_0 E$.

We will denote the coordinates of a point on the end of the unloading wave (UW) by $x = \lambda$ and $t = \theta$.

2. Solution with the Prandtl scheme[1]

At the instant of time $t = t_0$ two waves arise at the end of the rod, one of which propagates *a* the velocity *c* and the other with a velocity *kc*. When $t = T$, when the increase in the load on the rod $q(t)$ changes into a decrease $p(t)$, an unloading wave arises. Its end point lies on the wave which propagates with a velocity *kc*. Fig. 1 shows a sketch of the waves which illustrates the solution.

The differential equations of the characteristics and the conditions on them behind the UW (above UW in Fig. 1) have the form

$$dx = \pm c dt, \quad d\sigma \mp \rho c d\upsilon = 0 \tag{2.1}$$

In the region between the characteristic $x = kc(t - t_0)$ and the UW

$$dx = kc dt, \quad \sigma = q(t - x/(kc)), \quad \sigma = -k\rho c\upsilon + (1-k)\varepsilon_0 E \tag{2.2}$$

We will solve the problem of finding the unloading wave by representing the unloading wave by a section of a straight line

$$x = w(t - T), \quad w = \lambda/(\theta - T) \tag{2.3}$$

We will take the following characteristics (Fig. 1)

$$x = c(t - T_1), \quad x = kc(t - t_0), \quad x = -c(t - T_1), \quad x = kc(t - t_1)$$

Their intersections with the unloading wave (2.3) at the points (λ_1, θ_1), (λ, θ) give the equations

$$w(\theta - T) = c(\theta - T_1) = kc(\theta - t_0), \quad w(\theta_1 - T) = -c(\theta_1 - T_1) = kc(\theta_1 - t_1)$$

converting which we obtain

$$T_1 = \frac{w[(1-k)T + kt_0] - kct_0}{w - kc}, \quad t_1 = \frac{w(2T - t_0) + ct_0}{w + c} \tag{2.4}$$

[1] TARABRIN, G.T. Polynomial approximation of an unloading wave. Deposited at VINITI 17 June 1993. No.1993-V92.

Integrating differential relations (2.1) along the characteristics $x = c(t - T_1)$ and $x = -c(t - T_1)$, we obtain two equations

$$\sigma(\lambda, \theta) - \sigma(0, T_1) - \rho c[\upsilon(\lambda, \theta) - \upsilon(0, T_1)] = 0$$
$$\sigma(\lambda_1, \theta_1) - \sigma(0, T_1) + \rho c[\upsilon(\lambda_1, \theta_1) - \upsilon(0, T_1)] = 0$$

Adding these and taking into account the fact that

$$\sigma(\lambda, \theta) = \varepsilon_0 E, \quad \sigma(\lambda_1, \theta_1) = q(t_1), \quad \sigma(0, T_1) = p(T_1)$$

$$k\rho c\upsilon(\lambda_1, \theta_1) = -[q(t_1) - (1-k)\varepsilon_0 E], \quad \rho c\upsilon(\lambda, \theta) = -\varepsilon_0 E$$

we arrive at the equation

$$(1-k)q(t_1) + 2kp(T_1) = (1+k)\varepsilon_0 E \tag{2.5}$$

Eliminating t_1 and T_1 in this equation using expressions (2.4) we obtain an equation with a single unknown w, namely, the velocity of the unloading wave. By calculating w from formulae (2.3) we can also write an equation for the unloading wave and calculate the coordinates of its end λ and θ.

For a load which varies linearly

$$q(t) = \varepsilon_0 Et/t_0, \quad p(t) = \varepsilon_0 ET(T_0 - t)/[t_0(T_0 - T)] \tag{2.6}$$

the system of equations (2.3), (2.5) enables us to obtain the finished formulae

$$w = kc\sqrt{\frac{T_0}{T_0 - (1-\alpha)T}}, \quad \lambda = \frac{kcw(T - t_0)}{w - kc}, \quad \theta = T + \frac{\lambda}{w} \tag{2.7}$$

We can similarly solve the problem of the unloading wave when it is approximated by an algebraic polynomial of any finite order. For example, if we take the equation of the unloading wave in the form

$$x = w(t-T) + w_1(t-T)^2/2$$

where w and w_1 are unknown constants, then, when repeating the algorithm of the solution described, we must use the wave scheme with the characteristics represented by the dashed curve in Fig. 1.

A comparison of the results of actual numerical calculations enables us to conclude that, in the majority of cases, a quadratic approximation of the unloading wave is sufficient for practical purposes.

3. Solution with variable velocities of the strain waves

The relation $\sigma(\varepsilon)$ when ε increases beyond the elastic limit ε_0 is given by formula (1.1), and when ε decreases it is given by formula (1.2).

3.1. The loading region

In the region bounded by the unloading wave and the characteristic $x = kc(t - t_0)$ (Fig. 2), we have the following functional equation

$$\sigma = q(t - x/(cg(\varepsilon))) \tag{3.1}$$

Together with Eq. (1.1) and formula (1.5) it forms a system of equations, the solution of which determines σ and υ in this region.

The velocity is expressed in terms of the strain by the formula

$$\upsilon = -c\left[\varepsilon_0 + \int_{\varepsilon_0}^{\varepsilon} g(\varepsilon)d\varepsilon\right] = -c\varepsilon_0\left[1 + \frac{\beta^2}{3(\alpha\beta+1)}(k^3 - g^3(\varepsilon))\right] \tag{3.2}$$

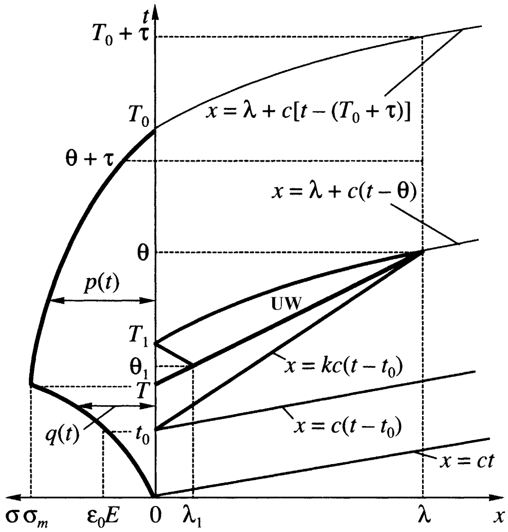

Fig. 2.

According to the definition of an unloading wave, the stress and strain at its given point (x, t), $x\in[0, \lambda]$ and $t\in[T, \theta]$ are greatest during the whole time the stress acts. It was previously agreed to denote the greatest strain in this section by e, and hence $e=e(x)$ is the strain in the unloading wave. When the load $q(t)$ increases monotonically the function $e(x)$ is monotonically decreasing. Its values fall from a maximum of ω_m on the end at the beginning of the unloading wave to the elastic limit e_0 at the end of the unloading wave when $x=\lambda$. The final point of the unloading wave (λ, θ) lies on the characteristic $x=kc(t-t_0)$.

3.2. The unloading region

When there is an unloading wave, $e(x)$, $s(x)=s(e(x))$ are known functions and the known function $z=cs(x)$ is the velocity of the waves in the unloading region. It follows from the inequalities

$$e'(x)<0,\quad s'(e)<0\Rightarrow s'(x)=s'(e)e'(x)>0$$

that $z=cs(z)$ is a monotonically increasing function. This means that in the region above the unloading wave the characteristics of the positive (negative) direction are convex (concave) curves.

When $x\geq\lambda$ the velocity of the strain waves is equal to c and the characteristics will be straight lines with slopes $\pm1/c$. Consequently, the characteristic curves in the region above the unloading wave on the straight line $x=\lambda$ will gradually join the straight-line characteristics in the region $x>\lambda$.

We will call the time the wave takes to traverse the section $[0, \lambda]$ with variable velocity $cs(x)$ the unloading period, and we will denote it by τ.

We assume that the loading wave has been constructed. Along the straight line $x=0$ we are given $\sigma(0,t)=p(t)$, while the function $\upsilon(0, t)$ is unknown. The integral of relation (1.6) on the negative-direction characteristic, connecting the point $(0, t_2)$, where $t_2\in(T, \theta+\tau)$, with a point on the unloading wave, is the final equation with a single unknown $\upsilon(0, t_2)$ and enables it to be calculated for any value of $t_2\in(T, \theta+\tau)$. At points (x, t), $x\neq0$) between the unloading wave and the characteristic connecting the points $(0, \theta+\tau)$, (λ,θ), the unknowns $\sigma(x, t)$, $\upsilon(x, t)$ are calculated from the solution of the system of two final equations with two unknowns. They are integrals of the relations (1.6) along the characteristics, starting from the point (x, t) and connecting this point with the t axis and the unloading wave.

Above the characteristic between the points $(0, \theta+\tau)$, (λ, θ) along the characteristic of the positive direction, σ and υ remain equal to their values at the points of the given characteristic and the straight line $x=0$ when $t>\theta+\tau$.

3.3. Construction of the unloading wave

The coefficient $s(x)$ is a monotonically increasing function, taking values from $s_0 = s(0)$ for the maximum strain ε_m to $1 = s(\lambda)$ for a strain equal to the elastic limit ε_0.

The problem is solved with simplifications which enable us to obtain the equations of the characteristics and the variation of the stress on them in explicit form in the region of unloading of the rod (in Fig. 2 this region is situated above the unloading wave UW).

It is assumed that the slope of the tangent to the characteristics varies linearly from $1/(cs_0)$ at $x=0$ to $1/c$ at $x=\lambda$. This reduces the differential equations of the characteristics (1.6) to the form

$$\frac{dt}{dx} = \pm\omega(x), \quad \omega(x) = \frac{1}{c}\left[1+\left(\frac{1}{s_0}-1\right)\left(1-\frac{x}{\lambda}\right)\right]$$

The general solution of these equations defines two families of parabolas

$$t = \pm W(x) + C, \quad W(x) = \frac{\lambda}{c}\left[\frac{x}{\lambda} - \frac{1}{2}\left(\frac{1}{s_0}-1\right)\left(1-\frac{x}{\lambda}\right)^2\right], \quad C = \text{const} \tag{3.3}$$

The differential relations on them (1.6) take the form

$$\omega(x)d\sigma \mp \rho dv = 0 \tag{3.4}$$

It is assumed that the stress varies linearly along the characteristic (3.3). Hence, if $M(0, T_1)$, $N(\xi, \eta)$ are points on the characteristic $t = t(x)$, i.e. $t(0) = T_1$ and $t(\xi) = \eta$, we have

$$\sigma(x, t(x)) = \sigma(0, T_1) - [\sigma(0, T_1) - \sigma(\xi, \eta)]x/\xi$$

Correspondingly we have

$$\int_{MN} \omega(x)d\sigma = -[\sigma(0, T_1) - \sigma(\xi, \eta)]\Delta(\xi); \quad \Delta(x) = [W(x) - W(0)]/x \tag{3.5}$$

We will solve the problem of constructing the unloading wave by representing it by a section of a straight line (2.3).

The scheme of the waves (Fig. 2) repeats the solution with the Prandtl diagram. The difference is that the characteristics above the unloading wave are the parabolas (3.3), while below the unloading wave only a single characteristic is taken, namely, $s = kc(t - t_0)$ By writing the equations of the characteristics and obtaining equations defining the coordinates of the points where they intersect with the unloading wave (2.3), we have

$$\begin{aligned} &\lambda = kc(\theta - t_0), \quad \theta = T_1 + \lambda\Delta(\lambda) \\ &\lambda_1/\lambda = (\theta_1 - T)/(\theta - T), \quad \theta_1 = T_1 - \lambda_1\Delta(\lambda_1) \end{aligned} \tag{3.6}$$

Integrating the differential relations (3.4) on sections of the characteristics with a common origin at the point $(0, T_1)$ and ends at the points (λ_1, θ_1), (λ, θ), adding the results of the integration, taking formula (3.5) into account and also taking into account the fact that

$$\sigma(\lambda, \theta) = \varepsilon_0 E, \quad v(\lambda, \theta) = -c\varepsilon_0, \quad \sigma(0, T_1) = p(T_1)$$

we obtain

$$p(T_1)[\Delta(\lambda) + \Delta(\lambda_1)] - \sigma(\lambda_1, \theta_1)\Delta(\lambda_1) - \varepsilon_0 E\Delta(\lambda) + \rho[c\varepsilon_0 + v(\lambda_1, \theta_1)] = 0 \tag{3.7}$$

Relations (1.1), (1.5), (3.1), (3.2), (3.6) and (3.7) represent a system of nine non-linear equations with nine unknowns:

$$\lambda, \theta, \lambda_1, \theta_1, T_1, \sigma(\lambda_1, \theta_1), v(\lambda_1, \theta_1), \varepsilon(\lambda_1, \theta_1), g(\lambda_1, \theta_1)$$

of which only two, λ and θ, are important.

Table 1

β		5	7	9	5	7	9	5	7	9
A	α	0.25			0.49			0.81		
	λ, cm	1.65	2.91	4.11	8.65	14.2	19.8	58.1	89.3	122
	$\theta \cdot 10^6$, s	16	21	25	32	47	61	127	190	257
B	α	0.17	0.16	0.16	0.32	0.31	0.31	0.53	0.53	0.53
	λ, cm	3.14	5.07	7.11	13.8	21.0	28.3	48.3	70.2	92.3
	$\theta \cdot 10^6$, s	24	33	48	54	78	103	131	188	246

The method of constructing the unloading wave in the form of the straight line (2.3) considered above can be extended to the case where the unloading wave is approximated by an algebraic polynomial of any positive integer degree.

Table 1 shows the results of calculations of λ and θ for a rectilinear unloading wave for the same materials, the strains of which are described by different $\sigma(\varepsilon)$ relations: A) relations (1.1) and (1.2) and B) the Prandtl diagram, which is a linearization of relations (1.1) and (1.2). The calculations were carried out for the following values of the constants

$$E = 2.1 \cdot 10^6 \text{ kg/cm}^2, \quad c = 5.5 \cdot 10^5 \text{ cm/s}, \quad \varepsilon_0 = (4/3) \cdot 10^{-3}, \quad t_0 = 10^{-5} \text{ s},$$
$$\beta = 5, 7, 9$$

where in the calculations of series A, $\alpha = 0.25$, 0.49 and 0.81, and the calculations of series B were carried out using formulae (2.7), in which the coefficient α, as is usually done when the actual $\sigma(\varepsilon)$ diagram is replaced by the Prandtl scheme, was calculated using the formula

$$\alpha = [(\sigma_m - \sigma_0)/(\varepsilon_m - \varepsilon_0)]/E$$

The load on the rod was assumed to vary linearly as given by (2.6), when the overall period $T_0 = 4T$, where T is the period during which the load increases to the maximum $q(T) = \sigma_m$, corresponding to the strain $\varepsilon_m = 0.8\varepsilon_*$.

In the calculations of series A, the solution of the system of equations was obtained by step-by-step approximation to the unknown T_1 using the method of simple iterations inside each approximation. All the results were obtained with a relative error with respect to the norm $l = \sqrt{\lambda^2 + [kc(\theta - T)]^2}$ of the order of 10^{-3}. Depending on the input data, the discrepancy (the absolute value of the ratio of the difference in the values of T_1 at the final and preceding steps to the value of T_1 at the preceding step) lay in the range from 0.11 to 0.19, where, beginning with a certain value of the step T_1, a reduction in the step ceased to affect the value of the discrepancy in the error. This can be both a consequence of the imperfection of the method of solving this system of equations and also an indication that the system of equations being solved is imperfect as a result of representing the unloading was by a section of a straight line and the use of other simplifications.

The results of the calculations of series A and B, obtained using different methods to described the relation $\sigma(\varepsilon)$ for the same material, differ considerably, which indicates the need to use functions $\sigma(\varepsilon)$ that most adequately describe the strain states of actual materials and, consequently, the suitability of the proposed method.

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